Exercise 46

Use logarithmic differentiation to find the derivative of the function.

$$y = \sqrt{x}^{\,x}$$

Solution

Take the natural logarithm of both sides and use the properties of logarithms to simplify the right side.

$$\begin{aligned}
\ln y &= \ln \sqrt{x}^{\,x} \\
&= x \ln \sqrt{x} \\
&= x \ln x^{1/2} \\
&= \frac{x}{2} \ln x
\end{aligned}$$

Differentiate both sides with respect to x.

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left(\frac{x}{2} \ln x\right) \\
\frac{1}{y} \cdot \frac{d}{dx}(y) &= \left[\frac{d}{dx}\left(\frac{x}{2}\right)\right] \ln x + \frac{x}{2}\left[\frac{d}{dx}(\ln x)\right] \\
\frac{1}{y} \cdot \frac{dy}{dx} &= \left(\frac{1}{2}\right) \ln x + \frac{x}{2}\left(\frac{1}{x}\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{\ln x}{2} + \frac{1}{2} \\
\frac{dy}{dx} &= y\left(\frac{\ln x + 1}{2}\right) \\
&= \sqrt{x}^{\,x}\left(\frac{\ln x + 1}{2}\right) \\
&= (x^{1/2})^x\left(\frac{\ln x + 1}{2}\right) \\
&= x^{x/2}\left(\frac{\ln x + 1}{2}\right) \\
&= \frac{x^{x/2}}{2}(\ln x + 1)
\end{aligned}$$